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A Criterion for the Nonuniqueness of the Measure of Orthogonality

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Let the polynomials $P_n(x)$, $n \geqslant 1$, be defned by $P_0(x) = 0$, $P_1(x) = 1$, $a_n P_{n+1}(x) + a_{n-1} P_{n-1}(x) + b_n P_n(x) = x P_n(x)$, $n \geqslant 1$. If $a_n > 0$ and b_n are real then there exists at least one measure of orthogonality for the polynomials $P_n(x)$, $n = 1, 2, \ldots$. The problem of finding conditions on the sequences a_n and b_n under which this measure is unique or nonunique still remains open for large classes of sequences a_n and b_n. Here a new criterion for the nonuniqueness of the measure of orthogonality is proved. This was achieved by proving that the infinite-dimensional Jacobi matrix associated with the sequences a_n and b_n is not self-adjoint.

1. INTRODUCTION

By measure of orthogonality we mean a probability measure μ on the Borel subsets of an interval $[\alpha, \beta]$ such that its support contains infinitely many points and the moments

$$\int_\alpha^\beta x^n \, d\mu, \qquad n \geqslant 0,$$

are finite in the case that the interval $[\alpha, \beta]$ is not bounded. To such a measure corresponds, up to a constant, a nondecreasing right continuous function $\mu(x)$ with infinitely many points at which it increases. The assumption that the increasing points are infinitely many is equivalent to the assertion

$$\int_\alpha^\beta P^2(x) \, d\mu > 0$$

for every polynomial P.

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The Gram–Schmidt orthogonalization procedure applied to the sequence 1, x, x^2, ..., x^n, ... in the Hilbert space $L^2(\mu)$ yields a three-term recurrence relation and a sequence of orthogonal polynomials

$$f_1 = 1, \qquad f_2 = (x - b_1) f_1,$$
$$f_{n+1} = (x - b_n) f_n - \frac{k_n^2}{k_{n-1}^2} f_{n-1}, \qquad n = 2, 3, ..., \tag{1.1}$$

where

$$k_n^2 = \| f_n \|^2 = \int_\alpha^\beta | f_n(x)|^2 \, d\mu, \qquad b_n = \frac{1}{k_n^2} \int_\alpha^\beta x \, | f_n(x)|^2 \, d\mu.$$

Thus the sequence $P_n(x) = k_1 f_n(x)/k_n$, $n = 1, 2, ...$, is a sequence of orthonormal polynomials which satisfy the recurrence relation

$$a_n P_{n+1}(x) + a_{n-1} P_{n-1}(x) + b_n P_n(x) = x P_n(x), \tag{1.2}$$

$$P_0(x) = 0, \qquad P_1(x) = 1, \qquad n = 1, 2, ..., \tag{1.3}$$

where

$$a_n = \frac{k_{n+1}}{k_n} > 0. \tag{1.3}$$

Conversely, given a sequence of polynomials, defined by (1.2) and (1.3) with $a_n > 0$ and b_n real, there exists at least one measure of orthogonality such that

$$\int_\alpha^\beta P_i(x) \, P_j(x) \, d\mu = \delta_{ij}.$$

An important problem in the theory of orthogonal polynomials is the problem of finding conditions on the sequences a_n and b_n such that the measure is unique. Difficulties appear in the case when a_n is unbounded.

It is well known that this uniqueness problem is equivalent to the uniqueness (determinary) of the power moment problem and that both problems are equivalent to the problem of finding conditions under which the Jacobi matrix T, defined by

$$Te_1 = a_1 e_2 + b_1 e_1,$$
$$Te_n = a_n e_{n+1} + a_{n-1} e_{n-1} + b_n e_n, \qquad n > 1, \tag{1.4}$$

is an (essentially) self-adjoint operator in an abstract separable Hilbert space H with the orthonormal basis e_n, $n \geqslant 1$. Among books and articles where these subjects and their relationships are treated we mention [1, 2, 6, 8–11]. For criteria concerning the self-adjointness of T (determinacy of the moment problem) see [1, 2, 7, 9, 10]. We mention the criterion of Carleman,

$$\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty,$$

and the criterion of Dennis and Wall,

$$\sum_{n=1}^{\infty} \frac{|b_{n+1}|}{a_n a_{n+1}} = \infty.$$

For the non-self-adjointness of T (indeterminary of the moment problem) the best known criterion (in terms of the sequences a_n and b_n) is the criterion of Berezanskii [2, p. 507] which holds under the conditions

(i) $|b_n| \leqslant M < \infty, \qquad n \geqslant 1,$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n} < \infty$

and for $n \geqslant n_0$ (iii) $a_{n+1} \cdot a_{n-1} \leqslant a_n^2$.

One of the disadvantages of this criterion is the assumption that the sequence b_n is bounded.

A relatively large number of criteria for both determinary and indeterminary concern particular cases with respect to the sequences a_n and b_n (see Refs. [3, 5] and the references presented therein). Many of these are based on the assumption that the sequence

$$\frac{a_n^2}{b_n b_{n+1}}, \qquad b_n > 0, \quad n \geqslant 1$$

is a chain sequence or equivalently that T is a positive operator [12]. This assumption restricts the spectrum of T to the semi-axis. In general, there still exist large classes of sequences a_n and b_n to which none of the known criteria can be applied.

Here we prove a new criterion for the non-self-adjointness of T which does not restrict the spectrum of T and is easy to handle in many cases where the criterion of Berezanskii is not applicable. The proof is based on a separation of T ($T = AV^* + VA + B$) in terms of simpler operators A, B, V, and V^*. Some results which we need about the operators AV^*, VA, and B are presented in Section 2. The main results (Theorem 3.1 and its corollaries) are presented in Section 3.

2. PRELIMINARY RESULTS AND LEMMAS

In an abstract complex Hilbert space H with the orthonormal basis e_n, $n \geqslant 1$, and scalar product (..) the operator (1.4) can be written as

$$T = AV^* + VA + B, \tag{2.1}$$

where A, B are the diagonal operators $Ae_n = a_n e_n$, $Be_n = b_n e_n$; V is the unilateral shift ($Ve_n = e_{n+1}$); and V^* is the adjoint of V ($V^* e_n = e_{n-1}$, $V^* e_1 = 0$), Usually T is defined on the linear manifold $D(T)$ consisting of finite linear combinations of the basis e_n. It is symmetric because $(Tx, y) = (x, Ty)$ for every x, y in $D(T)$ and $D(T)$ is dense in H. If the sequences a_n and b_n are bounded then the operators A and B are bounded; consequently T is a bounded operator on H and there is no problem with respect to the self-adjointness because $D(T) = D(T^*) = H$. Also, there is no problem in the case a_n is bounded and b_n is unbounded. Then T is also self-adjoint with $D(T) = D(B)$. A problem exists in the case where at least a_n is unbounded. In that case it is well known from the general theory of symmetric operators (see for instance [13, p. 108]) that T is not self-adjoint if and only if there exists an $x \neq 0$ in H such that

$$T^* x = ix \qquad (i^2 = -1). \tag{2.2}$$

In particular, for the operator (2.1) the following criterion [2, 7] is well known. If the series

$$\sum_{n=1}^{\infty} |P_n(i)|^2 \qquad (i^2 = -1) \tag{2.3}$$

converges (diverges), where $P_n(x)$ are the polynomials defined by (1.2) and (1.3), then T is not self-adjoint (is self-adjoint). The proof of the Berezanskii criterion is based on this more general criterion, which is not easy to handle.

The method which we follow here is based on the separation (2.1) of the Jacobian matrix T and requires a precise study of the definition domains of A, B, VA, and AV^*. With respect to the problem of self-adjointness of T we assume, without loss of generality, that $b_n \neq 0$ so that B has a bounded inverse. In the case a_n (b_n) is not bounded, A (B) has a self-adjoint extension to the range of the bounded and self-adjoint diagonal operator

$$A_0 \colon A_0 e_n = \frac{1}{a_n} e_n \qquad \left(B_0 \colon Be_n = \frac{1}{b_n} e_n \right), \qquad n \geqslant 1. \tag{2.4}$$

In fact, the spectrum of A_0 (B_0) consists of the eigenvalues of A_0 (B_0) and the point zero belongs to the essential spectrum of A_0 (B_0). Therefore the inverse A_0^{-1} (B_0^{-1}) of A_0 (B_0) exists and its definition domain $D(A_0^{-1}) = D(A) = (D(B_0^{-1}) = D(B))$ is dense in H. Thus $A = A_0^{-1}$ $(B = B_0^{-1})$ is an unbounded self-adjoint operator in H with its definition domain the range of the bounded operator A_0 (B_0). In other words,

$$D(A) = \{A^{-1}y, y \in H\} \qquad (D(B) = \{B^{-1}y, y \in H\}) \tag{2.5}$$

or

$$D(A) = \left\{ f \in H \colon \sum_{n=1}^{\infty} a_n^2 f_n^2 < \infty, f_n = |(f, e_n)| \right\} \tag{2.6}$$

$$\left(D(B) = \left\{ f \in H \colon \sum_{n=1}^{\infty} b_n^2 f_n^2 < \infty \right\} \right). \tag{2.7}$$

Sinve V is an isometry,

$$D(VA) = D(A). \tag{2.8}$$

The definition domain of AV^* is the set

$$D(AV^*) = \{x \in H \colon V^*x \in D(A)\} = \{x \in H \colon V^*x = A^{-1}y, y \in H\}. \tag{2.9}$$

Since $V^*V = I$ (identity operator) the equality $V^*x = A^{-1}y$ is equivalent to $V^*(x - VA^{-1}y) = 0$ or to $x = VA^{-1}y + ce_1$, in which c is a constant. Thus

$$D(AV^*) = \{x \in H \colon x = VA^{-1}y + ce_1, \ y \in H\}. \tag{2.10}$$

LEMMA 1. *Let C be the diagonal operator*

$$Ce_n = \frac{a_n}{a_{n+1}} e_n, \qquad n \geqslant 1, \tag{2.11}$$

where the sequence a_n/a_{n+1} is bounded. Then the bounded operator VC has the following properties:

$$VCf = A^{-1}VAf, \qquad f \in D(A) \tag{2.12}$$

$$(VC)^* h = AV^*A^{-1}h, \qquad h \in H. \tag{2.13}$$

Proof. For every f in $D(A)$ we have

$$A^{-1}VAf = \sum_{n=1}^{\infty} \frac{a_n}{a_{n+1}} (f, e_n) e_{n+1} = VCf.$$

This proves (2.12). Let h be an element of H. Then, due to (2.12), for every g in $D(A)$ we have

$$(g, (VC)^* h) = (VCg, h) = (A^{-1}VAg, h) = (VAg, A^{-1}h). \tag{2.14}$$

Since $A^{-1}h$ belongs to $D(A)$ and since the boundedness of a_n/a_{n+1} implies $D(A) \subseteq D(AV^*)$ we obtain from (2.14)

$$(g, (VC)^* h) = (g, AV^*A^{-1}h), \qquad g \in D(A). \tag{2.15}$$

Relation (2.13) follows from (2.15) because $D(A)$ is dense in H.

LEMMA 2. *Let the sequence $b_n/a_n(a_n/b_n)$ be bounded. Then $D(A) \subseteq D(B)$ $(D(B) \subseteq D(A))$ and the operator $A^{-1}B$ $(B^{-1}A)$ has an extension to the bounded self-adjoint and diagonal operator D (D^{-1}) defined by*

$$Df = \sum_{n=1}^{\infty} \frac{b_n}{a_n}(f, e_n)e_n, \qquad f \in H \tag{2.16}$$

$$\left(D^{-1}f = \sum_{n=1}^{\infty} \frac{a_n}{b_n}(f, e_n)e_n, \qquad f \in H\right). \tag{2.17}$$

Proof. $A^{-1}B$ $(B^{-1}A)$ is bounded on the dense linear manifold $D(B)$ $(D(A))$ and can therefore be extended to the bounded operator D (D^{-1}). The relation $D(A) \subseteq D(B)$ $(D(B) \subseteq D(A))$ follows from (2.6), (2.7), and the boundedness of b_n/a_n (a_n/b_n). Note that on $D(A) \cap D(B)$ the relation

$$AD = DA \ (BD^{-1} = D^{-1}B) \tag{2.18}$$

holds.

Remark. The definition domain $D(T)$ of $T = AV^* + VA + B$ is

$$D(T) = D(A) \cap D(B) \cap D(AV^*). \tag{2.19}$$

Thus

$$D(T) \subseteq D(A). \tag{2.20}$$

3. MAIN RESULTS

Assume that

$$\lim_{n\to\infty} a_n = \infty \tag{3.1}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \alpha \tag{3.2}$$

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \gamma \tag{3.3}$$

and denote by ϱ_i, $i = 1, 2$, the roots of the algebraic equation $\alpha x^2 + \gamma x + 1 = 0$.

THEOREM 3.1. *Let one of the conditions*

$$\begin{aligned} &\text{(i)} \quad \alpha \neq 0, \quad |\varrho_i| > 1, \quad i = 1, 2, \\ &\text{(ii)} \quad \alpha = 0, \quad |\gamma| < 1, \end{aligned} \tag{3.4}$$

be satisfied. Then the operator $T = AV^* + VA + B$ *is not self-adjoint.*

Note that $\alpha \geqslant 0$ and the condition (i) implies that $0 < \alpha < 1$.

Proof. We shall prove that there exists an element $x \neq 0$ in $D(T^*)$ such that $T^*x = ix$ (as noted in Secton 2).

Consider the bounded operator

$$V^* + VC + D - iA^{-1}, \tag{3.5}$$

where, due to (3.1), the operator A^{-1} is compact. The operator (3.5) can be written as

$$V^* + V(C - \alpha I) + \alpha V + (D - \gamma I) + \gamma I - iA^{-1},$$

where, due to (2.11), (3.2) and (2.16), (3.3), both operators $C - \alpha I$ and $D - \gamma I$ are compact. Thus (3.5) has the form

$$V^* + \alpha V + \gamma I + K$$

or

$$V^*[(\alpha V^2 + \gamma V + I) + VK], \tag{3.6}$$

where

$$K = V(C - \alpha I) + D - \gamma I - iA^{-1}. \tag{3.7}$$

Observe that VK is compact.

Since the spectrum of V is the entire closed unit disc, condition (i) or (ii) implies that the operator $\alpha V^2 + \gamma V + I = \alpha(V - \varrho_1 I)(V - \varrho_2 I)$ in case $\alpha \neq 0$ or the operator $I + \gamma V$ in case $\alpha = 0$ has a bounded inverse. Thus the operator (3.6) can be written as

$$V^*(\alpha V^2 + \gamma V + I)(I + \Delta), \tag{3.8}$$

where

$$\Delta = (\alpha V^2 + \gamma V + I)^{-1} VK \tag{3.9}$$

is compact. Now the Fredholm alternative [13, p. 136] implies either $(I+\Delta)x=0$, $x\neq 0$, or the operator $I+\Delta$ has a bounded inverse. Thus either x given by $(I+\Delta)x=0$ or

$$x=(I+\Delta)^{-1}(\alpha V^*+\gamma V+I)^{-1}e_1 \tag{3.10}$$

is a solution of the equation

$$V^*(\alpha V^2+\gamma V+I)(I+\Delta)x=0$$

or

$$(V^*+VC+D)x=iA^{-1}x, \qquad x\in H, \quad x\neq 0. \tag{3.11}$$

Since $A^{-1}x\in D(A)$ we obtain from (3.11)

$$A(V^*+VC+D)x=ix$$

and

$$(f, A(V^*+VC+D)x)=(f, ix), \qquad f\in D(T).$$

Thus by (2.20) and the self-adjointness of A,

$$(Af, V^*x+VCx+Dx)=(f, ix)$$

or

$$(Af, V^*x)+(Af, VCx)+(Af, Dx)=(f, ix)$$

and

$$(VAf, x)+((VC)^*Af, x)+(DAf, x)=(f, ix). \tag{3.12}$$

Now by Lemmas 1 and 2 we find from (3.12)

$$(VAf, x)+(AV^*f, x)+(DAf, x)=(f, ix)$$

or

$$(VAf, x)+(AV^*, x)+(Bf, x)=(f, ix)$$

and

$$(Tf, x)=(f, ix), \qquad f\in D(T). \tag{3.13}$$

The last relation means that $x\in D(T)$ and $T^*x=ix$.

Corollary 1. *Let* $a_n \to \infty$ *and let*

$$\gamma^2 \leqslant 4\alpha.$$

Then the condition $\alpha \geqslant 1$ *is necessary in order that the moment problem associated with* a_n *and* b_n *is determinate. In particular, the above condition is necessary in order that the moment problem corresponding to the symmetric polynomials* $a_n P_{n+1}(x) + a_{n-1} P_{n-1}(x) = x P_n(x)$, $P_0(x) = 0$, $P_1(x) = 1$, *is determinate.*

Proof. Suppose that the moment problem associated with a_n and b_n is determinate, i.e., $T = AV^* + VA + B$ is self-adjoint. Then $\alpha \geqslant 1$ must be satisfied because $\alpha < 1$ implies, by Theorem 3.1, non-self-adjointness of T.

Corollary 2. *Let* $a_n \to \infty$,

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = \alpha < 1,$$

and let the sequence b_n *be bounded. Then* $T = AV^* + VA + B$ *is not self-adjoint.*

Proof. Since b_n is bounded,

$$\gamma = \lim_{n \to \infty} \frac{b_n}{a_n} = 0,$$

and since $\alpha < 1$, the condition (i) of the theorem is satisfied.

Remark. A well known comparison theorem, due to Carleman (see Ref. [4]), states that "If the sequences $a_n - a'_n$ and $b_n - b'_n$ are bounded then the associated moment problem with the sequences a'_n, b'_n is determinate, provided the associated moment problem with the sequences a_n and b_n is determinate." In other words, the addition of bounded sequences to the sequences a_n and b_n leaves the determinacy of the moment problem invariant, since a self-adjoint operator perturbed by a bounded operator is also self-adjoint. This result of Carleman folows easily on the basis of equivalence of self-adjointness and determinacy of the moment problem. Also, we can easily proved the following:

Theorem 3.2. *Let the sequences* $a_n - a'_n$ *and* $b_n - b'_n$ *be bounded and let* $T = AV^* + VA + B$ *be non-self-adjoint. Then* $T' = A'V^* + VA' + B'$ *is also non-self-adjoint.*

Proof. Observe that $C = T - T' = (A - A')\,V^* + (A - A')\,V + B - B'$ is bounded. This means that self-adjointness of T' implies self-adjointness of T contrary to the assumption. Thus T' is not self-adjoint.

Putting together Theorems 3.1 and 3.2 we obtain

COROLLARY 3. *Let the conditions of Theorem* 3.1 *be satisfied and let* c_n, d_n *be bounded sequences. Then the moment problem associated with the sequences* $a_n + c_n$, $b_n + d_n$ *is indeterminate.*

EXAMPLE 1. The sequence of polynomials Q_n, $n \geqslant 1$, defined by

$$Q_1(x) = 1,$$
$$-xQ_1(x) = -(\lambda_1 + \mu_1)\,Q_1(x) + \lambda_1 Q_2(x),$$
$$-xQ_n(x) = \lambda_n Q_{n+1}(x) + \mu_n Q_{n-1}(x) - (\lambda_n + \mu_n)\,Q_n(x),$$

appears in the theory of birth and death processes [7].

The above recurrence relation, by setting

$$\tau_1 = \lambda_1$$
$$\tau_n = \frac{\lambda_1 \lambda_2 \cdots \lambda_n}{\mu_2 \mu_3 \cdots \mu_n}, \qquad n \geqslant 2,$$

and

$$Q_n(x) = (-1)^{n+1} \sqrt{\frac{\lambda_n}{\tau_n}}\, P_n(x), \qquad n \geqslant 2.$$

is transformed to the normalized form

$$\begin{gathered} \sqrt{\lambda_n \mu_{n+1}}\, P_{n+1}(x) + \sqrt{\lambda_{n-1} \mu_n}\, P_{n-1}(x) + (\lambda_n + \mu_n)\, P_n(x) = xP_n(x) \\ P_0(x) = 0, \qquad P_1(x) = 1, \qquad \lambda_n > 0, \qquad n \geqslant 1, \qquad \mu_n > 0, \qquad n \geqslant 2. \end{gathered} \tag{3.15}$$

It is well known [7, p. 528] that when $\mu_1 > 0$ the condition

$$\sum_{n=1}^{\infty} \pi_n Q_n^2(0) = \infty \tag{3.16}$$

is a necessary and sufficient condition that the moment problem associated with μ_n and λ_n is determinate.

In (3.1),

$$\pi_n = \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{\mu_2 \mu_3 \cdots \mu_n}, \qquad \pi_1 = 1$$

and

$$Q_n(0) = 1 + \mu_1 \sum_{j=1}^{n-1} \frac{1}{\lambda_j \pi_j}.$$

Now consider the following example for the birth and death processes:

$$\lambda_n = \mu_n = k^n, \qquad n \geqslant 1, \quad k > 1. \tag{3.17}$$

Then

$$\pi_n = \frac{1}{k^{n-1}}, \qquad \lambda_n \pi_n = k, \qquad Q_n(0) = n,$$

and the series in (3.16) takes the form

$$\sum_{n=1}^{\infty} \frac{n^2}{k^{n-1}},$$

which converges by the ratio test. Thus the above-mentioned criterion implies that the moment problem associated with the sequences (3.17) is indeterminate.

To apply our criterion we obtain from (3.15)

$$a_n = k^n \sqrt{k}, \qquad b_n = 2k^n, \qquad k > 1$$

and

$$\alpha = \lim_{n \to \infty} \frac{a_n}{a_{n+1}} = \frac{1}{k}, \qquad \gamma = \lim_{n \to \infty} \frac{b_n}{a_n} = \frac{2}{\sqrt{k}}.$$

The roots ϱ_1 and ϱ_2 of the equation $\alpha x^2 + \gamma x + 1 = 0$ are

$$|\varrho_1| = |\varrho_2| = \sqrt{k} > 1$$

and Theorem 3.1 implies the indeterminacy of the moment problem.

Remark. Of course, Theorem 3.1 does not apply to all well-known indeterminate problems. Also, it does not cover as a particular case the Berezanskii criterion. However, there exist large classes of sequences where Theorem 3.1 works and the Beresanskii criterion does not. We give below an example.

EXAMPLE 2. Let $a_n = n!$, $n = 1, 2, 3, \ldots$, and let b_n be an unbounded sequence of real numbers such that

$$\lim_{n \to \infty} \frac{b_n}{n!} = 0.$$

In this case our criterion predicts indeterminacy because condition (ii) of Theorem 3.1 is satisfied. The criterion of Berezanskii cannot be applied because the sequence b_n is unbounded.

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REFERENCES

2. Ju. M. Berezanskii, "Expansions in Eigenfunctions of Self-adjoint Operators," Amer. Math. Soc., Providence, RI, 1968.
3. T. S. Chihara, "An Introduction to Orthogonal Polynomials," Gordon and Breach, New York/London/Paris, 1978.
4. T. S. Chihara, Comparison theorems for Hambunger moment problems, *in* "Orthogonal Polynomials and Their Applications" (C. Brezinski, L. Gori, and A. Ronveaux, Eds.), pp. 17–20, Baltzer, 1991.
5. T. S. Chihara, Hamburger moment problem and orthogonal polynomials, *Trans. Amer. Math. Soc.* **315**, No. 1 (1989), 189–203.
6. D. Masson and W. K. McClary, Classes of C^∞ vectors and essential self-adjointness, *J. Funct. Anal.* **10** (1972), 19–32.
7. S. Karlin and J. L. McGregor, The differential equations of the birth and death processes and the Stieltjes moment problem, *Trans. Amer. Math. Soc.* **85** (1957), 489–546.
9. D. Sarason, Moment problems and operators in Hilbert space, *Proc. Sympos. Appl. Math.* **37** (1987), 54–70.
10. J. A. Shohat and J. D. Tamarkin, "The Problem of Moments," Mathematical Surveys, Vol. I, Amer. Math. Soc., Providence, RI, 1943.
11. A. Wouk, Difference equations and J matrices, *Duke Math. J.* **20** (1953), 141–159.
12. A. S. Wall and M. Wetzel, Quadratic forms and convergence relations for continued fractions, *Duke Math. J.* **11** (1944), 89–102.
13. J. Weidmann, "Linear Operators in Hilbert Space," Springer-Verlag, New York, 1980.